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NEUTROSOPHIC LIE SUBALGEBRAS AND IDEALS: IMAGES, INVERSE IMAGES, AND QUOTIENT STRUCTURES

MUKHTAR AHMAD[1)] AND FLORENTIN SMARANDACHE[2,*)]

[1)]Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia

[2)]University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

Abstract. In this paper, we study neutrosophic Lie subalgebras and neutrosophic Lie ideals under Lie algebra homomorphisms. We investigate the behavior of these structures through images and inverse images and establish several preservation properties related to Lie algebra morphisms. We also examine sums of neutrosophic Lie ideals, upper level subsets, and quotient structures associated with kernels of homomorphisms. This work extends known results tothe context of neutrosophic Lie algebras, contributing new insights into their behavior under algebraic mappings.

1. Introduction

The theory of uncertainty has become one of the most active and influential areas of modern mathematics due to its wide range of applications in algebra, engineering, artificial intelligence, information sciences, and decision-making problems. The pioneering work of [1] introduced the notion of fuzzy sets, which provided a mathematical framework for dealing with vagueness and imprecise information. Since then, fuzzy set theory has been extensively generalized and applied in many branches of mathematics and theoretical sciences.

To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

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mukhkhu0786@gmail.com[1]; 30125701@mahasiswa.itb.ac.id[1]; ORCID: 0009-0006-4597-2278
smarand@unm.edu[2]; ORCID: 0000-0002-5560-5926.

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1. Introduction

Motivated by these observations, the present paper investigates neutrosophic Lie subalgebras and neutrosophic Lie ideals through Lie algebra homomorphisms. We study the behavior of neutrosophic Lie structures under images and inverse images and establish several preservation properties. Further, we provide a neutrosophic framework for dealing with vagueness and imprecise information. Since its introduction by [1], neutrosophic set theory has been extensively generalized and applied in many branches of mathematics and theoretical sciences.

To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on the intuitionistic fuzzy framework were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy techniques in studying algebraic systems with uncertain behavior.

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Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of freedom makes neutrosophic theory highly suitable for dealing with incomplete and contradictory data. Later, [11] introduced single-valued neutrosophic sets, which simplified the practical implementation of neutrosophic concepts and accelerated their applications in several mathematical disciplines.

In recent years, neutrosophic algebraic structures have attracted significant research interest. Various studies on neutrosophic groups, rings, modules, semigroups, and BCK/BCI-algebras have been carried out by many researchers [12, 13, 14, 15, 16, 17, 18]. The incorporation of neutrosophic ideas into algebraic systems has led to deeper insights into the role of uncertainty and indeterminacy in algebraic operations and homomorphic structures.

Lie algebras constitute one of the central topics in modern algebra due to their profound connections with geometry, topology, differential equations, and theoretical physics. They play a fundamental role in quantum mechanics, particle physics, symmetry theory, and representation theory [19, 20]. Because of their importance, fuzzy and intuitionistic fuzzy extensions of Lie algebras have been studied extensively over the last few decades [21, 22, 23]. These investigations focused mainly on fuzzy Lie subalgebras, fuzzy Lie ideals, intuitionistic fuzzy Lie structures, and their homomorphic properties.

The study of homomorphisms in fuzzy and neutrosophic algebraic structures is particularly important because homomorphisms preserve essential algebraic information between structures. Several authors have investigated images, inverse images, quotient structures, and level subsets in fuzzy and intuitionistic fuzzy algebraic systems [24, 25, 26]. However, the theory of neutrosophic Lie ideals and neutrosophic Lie subalgebras under Lie algebra homomorphisms remains comparatively underdeveloped.

2. Fundamentals of Neutrosophic Lie Ideals

A neutrosophic set defined on a non-empty set X is denoted by $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$, where $\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E} : X \to [0,1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. For every $\mathbf{x} \in X$, the functions satisfy $0 \leq \mathbf{T_E}(\mathbf{x}) + \mathbf{I_E}(\mathbf{x}) + \mathbf{F_E}(\mathbf{x}) \leq 3$. The concept of neutrosophic sets generalizes fuzzy sets and intuitionistic fuzzy sets. In particular:

(1) if $\mathbf{I_E}(\mathbf{x}) = 0$, then $\mathbf{E}$ reduces to an intuitionistic fuzzy set;
(2) if additionally $\mathbf{F_E}(\mathbf{x}) = 1 - \mathbf{T_E}(\mathbf{x})$, then $\mathbf{E}$ becomes a classical fuzzy set.

Let Ξ be a Lie algebra over a field $\mathbb{F}$. A neutrosophic set $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$ on Ξ is called a neutrosophic Lie ideal of Ξ if, for all $\mathbf{x}, ß \in \Xi$ and $\alpha \in \mathbb{F}$, the following conditions hold:

(1) $\mathbf{T_E}(\mathbf{x} + ß) \geq \min\{\mathbf{T_E}(\mathbf{x}), \mathbf{T_E}(ß)\}$,
(2) $\mathbf{I_E}(\mathbf{x} + ß) \geq \min\{\mathbf{I_E}(\mathbf{x}), \mathbf{I_E}(ß)\}$,
(3) $\mathbf{F_E}(\mathbf{x} + ß) \leq \max\{\mathbf{F_E}(\mathbf{x}), \mathbf{F_E}(ß)\}$,
(4) $\mathbf{T_E}(\alpha\mathbf{x}) \geq \mathbf{T_E}(\mathbf{x})$,
(5) $\mathbf{I_E}(\alpha\mathbf{x}) \geq \mathbf{I_E}(\mathbf{x})$,

(6) $\mathbf{F_E}(\alpha \mathbf{x}) \leq \mathbf{F_E}(\mathbf{x})$,
(7) $\mathbf{T_E}([\mathbf{x}, ß]) \geq \min\{\mathbf{T_E}(\mathbf{x}), \mathbf{T_E}(ß)\}$,
(8) $\mathbf{I_E}([\mathbf{x}, ß]) \geq \min\{\mathbf{I_E}(\mathbf{x}), \mathbf{I_E}(ß)\}$,
(9) $\mathbf{F_E}([\mathbf{x}, ß]) \leq \max\{\mathbf{F_E}(\mathbf{x}), \mathbf{F_E}(ß)\}$.

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Definition 2.1. *Let $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$ be a neutrosophic set defined on a Lie algebra Ξ over a field $\mathbb{F}$. Then, $\mathbf{E}$ is called a neutrosophic Lie subalgebra of Ξ if, for all $\mathbf{x}, ß \in \Xi$ and $\alpha \in \mathbb{F}$, the following conditions hold:*

(1)

$$\mathbf{T_E}(\mathbf{x} + ß) \geq \mathbf{T_E}(\mathbf{x}) \wedge \mathbf{T_E}(ß),$$
$$\mathbf{I_E}(\mathbf{x} + ß) \geq \mathbf{I_E}(\mathbf{x}) \wedge \mathbf{I_E}(ß),$$
$$\mathbf{F_E}(\mathbf{x} + ß) \leq \mathbf{F_E}(\mathbf{x}) \vee \mathbf{F_E}(ß);$$

(2)

$$\mathbf{T_E}(\alpha \mathbf{x}) \geq \mathbf{T_E}(\mathbf{x}),$$
$$\mathbf{I_E}(\alpha \mathbf{x}) \geq \mathbf{I_E}(\mathbf{x}),$$
$$\mathbf{F_E}(\alpha \mathbf{x}) \leq \mathbf{F_E}(\mathbf{x});$$

(3)

$$\mathbf{T_E}([\mathbf{x}, ß]) \geq \mathbf{T_E}(\mathbf{x}) \wedge \mathbf{T_E}(ß),$$
$$\mathbf{I_E}([\mathbf{x}, ß]) \geq \mathbf{I_E}(\mathbf{x}) \wedge \mathbf{I_E}(ß),$$
$$\mathbf{F_E}([\mathbf{x}, ß]) \leq \mathbf{F_E}(\mathbf{x}) \vee \mathbf{F_E}(ß).$$

If condition (3) is replaced by

$$\mathbf{T_E}([\mathbf{x}, ß]) \geq \mathbf{T_E}(\mathbf{x}) \vee \mathbf{T_E}(ß),$$
$$\mathbf{I_E}([\mathbf{x}, ß]) \geq \mathbf{I_E}(\mathbf{x}) \vee \mathbf{I_E}(ß),$$
$$\mathbf{F_E}([\mathbf{x}, ß]) \leq \mathbf{F_E}(\mathbf{x}) \wedge \mathbf{F_E}(ß),$$

Then, $\mathbf{E}$ is called a neutrosophic Lie ideal of Ξ.

3. Images and Inverse Images of Neutrosophic Lie Subalgebras (Ideals) under Lie Algebra Homomorphisms

Let Ξ_1 and Ξ_2 be Lie algebras, let $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$ be a neutrosophic subset of Ξ_1, and let $\aleph : \Xi_1 \to \Xi_2$ be a mapping. The neutrosophic subset $\aleph(\mathbf{E}) = (\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})})$ of $\aleph(\Xi_1)$ is defined by, for every $ß \in \aleph(\Xi_1)$,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß) = \sup_{\mathbf{x} \in \aleph^{-1}(\{ß\})} \{\mathbf{T_E}(\mathbf{x})\},$$
$$\mathbf{I}_{\aleph(\mathbf{E})}(ß) = \sup_{\mathbf{x} \in \aleph^{-1}(\{ß\})} \{\mathbf{I_E}(\mathbf{x})\},$$
$$\mathbf{F}_{\aleph(\mathbf{E})}(ß) = \inf_{\mathbf{x} \in \aleph^{-1}(\{ß\})} \{\mathbf{F_E}(\mathbf{x})\}.$$

The neutrosophic subset $\aleph(\mathbf{E})$ is called the image (mapping) of $\mathbf{E}$ under $\aleph$.

Similarly, let $\mathsf{P} = (\mathbf{T_P}, \mathbf{I_P}, \mathbf{F_P})$ be a neutrosophic subset of Ξ_2. The inverse image of P under $\aleph$, denoted by $\aleph^{-1}(\mathsf{P})$, is defined as $\aleph^{-1}(\mathsf{P}) = \left(\mathbf{T}_{\aleph^{-1}(\mathsf{P})}, \mathbf{I}_{\aleph^{-1}(\mathsf{P})}, \mathbf{F}_{\aleph^{-1}(\mathsf{P})}\right)$, where for every $\mathbf{x} \in \Xi_1$,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T_P}(\aleph(\mathbf{x})),$$
$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{I_P}(\aleph(\mathbf{x})),$$

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and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})).$$

Theorem 3.1. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism. If $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ is a neutrosophic Lie subalgebra of Ξ_2, then the inverse image $\aleph^{-1}(\mathsf{P})$ is a neutrosophic Lie subalgebra of Ξ_1.*

Proof. To prove that $\aleph^{-1}(\mathsf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathsf{P})}, \mathbf{I}_{\aleph^{-1}(\mathsf{P})}, \mathbf{F}_{\aleph^{-1}(\mathsf{P})})$ is a neutrosophic Lie subalgebra of Ξ_1, we verify the defining conditions. Let $\mathbf{x}, ß \in \Xi_1$ and $\alpha \in \mathbb{F}$. Since

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})), \ \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x})), \text{ and } \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})),$$

we proceed as follows.

(i) Addition: Since $\aleph$ is linear,

$$\aleph(\mathbf{x} + ß) = \aleph(\mathbf{x}) + \aleph(ß).$$

Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) = \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x} + ß)) = \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x}) + \aleph(ß)).$$

Because P is a neutrosophic Lie subalgebra of Ξ_2,

$$\mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x}) + \aleph(ß)) \geq \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{T}_{\mathsf{P}}(\aleph(ß)).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) \geq \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) = \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x} + ß)) = \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x}) + \aleph(ß))$$
$$\geq \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathsf{P}}(\aleph(ß)),$$

so

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) \geq \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Also,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) = \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x} + ß)) = \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x}) + \aleph(ß))$$
$$\leq \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})) \vee \mathbf{F}_{\mathsf{P}}(\aleph(ß)),$$

and therefore

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x} + ß) \leq \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \vee \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(ß).$$

(ii) Scalar Multiplication: For any $\alpha \in \mathbb{F}$,

$$\aleph(\alpha\mathbf{x}) = \alpha\aleph(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\alpha\mathbf{x}) = \mathbf{T}_{\mathsf{P}}(\aleph(\alpha\mathbf{x})) = \mathbf{T}_{\mathsf{P}}(\alpha\aleph(\mathbf{x})) \geq \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})),$$

which implies

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\alpha\mathbf{x}) \geq \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\alpha\mathbf{x}) = \mathbf{I}_{\mathsf{P}}(\alpha\aleph(\mathbf{x})) \geq \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x}))$$

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Also,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\alpha\mathbf{x}) = \mathbf{F}_{\mathsf{P}}(\alpha\aleph(\mathbf{x})) \leq \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})).$$

So,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\alpha\mathbf{x}) \leq \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}).$$

(iii) Lie Bracket: Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph([\mathbf{x}, ß]) = [\aleph(\mathbf{x}), \aleph(ß)].$$

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Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{T}_{\mathsf{P}}(\aleph([\mathbf{x},ß])) = \mathbf{T}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]).$$

Since P is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \geq \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \geq \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{I}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]) \geq \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathsf{P}}(\aleph(ß)).$$

Thus,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \geq \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Also,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{F}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]) \leq \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})) \vee \mathbf{F}_{\mathsf{P}}(\aleph(ß)),$$

which implies

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \leq \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \vee \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Therefore, $\aleph^{-1}(\mathsf{P})$ satisfies all the conditions of a neutrosophic Lie subalgebra of Ξ_1. □

Corollary 3.1. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism. If $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ is a neutrosophic Lie ideal of Ξ_2. Then, the inverse image $\aleph^{-1}(\mathsf{P})$ is also a neutrosophic Lie ideal of Ξ_1.*

Proof. The proof follows similarly to Theorem 1. We only verify the Lie ideal condition involving the Lie bracket.

Let $\mathbf{x}, ß \in \Xi_1$. Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph([\mathbf{x},ß]) = [\aleph(\mathbf{x}),\aleph(ß)].$$

And the neutrosophic function, we have

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{T}_{\mathsf{P}}(\aleph([\mathbf{x},ß])) = \mathbf{T}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]).$$

Since P is a neutrosophic Lie ideal of Ξ_2,

$$\mathbf{T}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]) \geq \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})) \vee \mathbf{T}_{\mathsf{P}}(\aleph(ß)).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \geq \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \vee \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Similarly for the indeterminacy-membership function,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{I}_{\mathsf{P}}(\aleph([\mathbf{x},ß])) = \mathbf{I}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]) \geq \mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathsf{P}}(\aleph(ß)),$$

and therefore

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \geq \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(ß).$$

For the falsity-membership function,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) = \mathbf{F}_{\mathsf{P}}(\aleph([\mathbf{x},ß])) = \mathbf{F}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]).$$

Since P is a neutrosophic Lie ideal,

$$\mathbf{F}_{\mathsf{P}}([\aleph(\mathbf{x}),\aleph(ß)]) \leq \mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{F}_{\mathsf{P}}(\aleph(ß)).$$

Thus,

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}([\mathbf{x},ß]) \leq \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \wedge \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(ß).$$

Therefore, $\aleph^{-1}(\mathsf{P})$ satisfies all the conditions of a neutrosophic Lie ideal of Ξ_1. □

Theorem 3.2. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ is a neutrosophic Lie subalgebra of Ξ_1. Then, the image $\aleph(\mathbf{E})$ is a neutrosophic Lie subalgebra of $\aleph(\Xi_1)$.*

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Abstract. In this paper, we study neutrosophic Lie subalgebras and neutrosophic Lie ideals under Lie algebra homomorphisms. We investigate the behavior of these structures through images and inverse images and establish several preservation properties related to Lie algebra morphisms. We also examine sums of neutrosophic Lie ideals, quotient structures associated with kernels of homomorphisms. This work extends known results tothe context of neutrosophic Lie algebras, contributing new insights into their behavior under algebraic mappings.

1. Introduction

The theory of uncertainty has become one of the most active and influential areas of modern mathematics due to its wide range of applications in algebra, engineering, artificial intelligence, information sciences, and decision-making problems. The pioneering work of [1] introduced the notion of fuzzy sets, which provided a mathematical framework for dealing with vagueness and imprecise information. Since then, fuzzy set theory has been extensively generalized and applied in many branches of mathematics and theoretical sciences.

To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

Proof. First, we show that $\aleph(\mathbf{E}) = \left(\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})}\right)$ is a neutrosophic set on $\aleph(\Xi_1)$. For every $ß \in \aleph(\Xi_1)$,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß) = \sup_{ß=\aleph(\mathbf{x})} \{\mathbf{T}_{\mathbf{E}}(x)\}, \quad \mathbf{I}_{\aleph(\mathbf{E})}(ß) = \sup_{ß=\aleph(\mathbf{x})} \{\mathbf{I}_{\mathbf{E}}(x)\}, \quad \text{and} \quad \mathbf{F}_{\aleph(\mathbf{E})}(ß) = \inf_{ß=\aleph(\mathbf{x})} \{\mathbf{F}_{\mathbf{E}}(x)\}.$$

Next, we verify the defining conditions of a neutrosophic Lie subalgebra.

Let $ß_1, ß_2 \in \aleph(\Xi_1)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathbf{x}_1) = ß_1 \quad \text{and} \quad \aleph(\mathbf{x}_2) = ß_2.$$

(i) Since $\aleph$ is linear,

$$ß_1 + ß_2 = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2) = \aleph(\mathbf{x}_1 + \mathbf{x}_2).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß_1 + ß_2) = \sup_{ß_1+ß_2=\aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2).$$

Since $\mathbf{E}$ is a neutrosophic Lie subalgebra of Ξ_1,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(ß_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(ß_2),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(ß_2).$$

(ii) Let $\alpha \in \mathbb{F}$. Since

$$\alpha ß_1 = \alpha\aleph(\mathbf{x}_1) = \aleph(\alpha\mathbf{x}_1),$$

we obtain

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha ß_1) = \sup_{\alpha ß_1=\aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1).$$

Since $\mathbf{E}$ is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1).$$

Thus,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha ß_1) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(\alpha ß_1) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\alpha ß_1) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1).$$

(iii) Since $\aleph$ is a Lie algebra homomorphism,

$$[ß_1, ß_2] = [\aleph(\mathbf{x}_1), \aleph(\mathbf{x}_2)] = \aleph([\mathbf{x}_1, \mathbf{x}_2]).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \sup_{[ß_1,ß_2]=\aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Because $\mathbf{E}$ is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(ß_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(ß_2),$$

NEUTROSOPHIC LIE SUBALGEBRAS AND IDEALS: IMAGES, INVERSE IMAGES, AND QUOTIENT STRUCTURES

MUKHTAR AHMAD[1)] AND FLORENTIN SMARANDACHE[2,*)]

[1)]Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia

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and

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(ß_2).$$

Therefore, $\aleph(\mathbf{E})$ satisfies all the conditions of a neutrosophic Lie subalgebra of $\text{im}(\aleph)$. □

Corollary 3.2. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$ is a neutrosophic Lie ideal of Ξ_1, then the image $\aleph(\mathbf{E})$ is a neutrosophic Lie ideal of* $\text{im}(\aleph)$.

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Proof. It is sufficient to verify the Lie ideal conditions for the Lie bracket.

Let $ß_1, ß_2 \in \text{im}(\aleph)$. Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathbf{x}_1) = ß_1 \quad \text{and} \quad \aleph(\mathbf{x}_2) = ß_2.$$

Since $\aleph$ is a Lie algebra homomorphism, we have

$$[ß_1, ß_2] = [\aleph(\mathbf{x}_1), \aleph(\mathbf{x}_2)] = \aleph([\mathbf{x}_1, \mathbf{x}_2]),$$

and,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \sup_{[ß_1, ß_2] = \aleph(z)} \{\mathbf{T_E}(z)\}.$$

In particular,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{T_E}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ_1,

$$\mathbf{T_E}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T_E}(\mathbf{x}_1) \vee \mathbf{T_E}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{T}_{\aleph(\mathbf{E})}(ß_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \sup_{[ß_1, ß_2] = \aleph(z)} \{\mathbf{I_E}(z)\} \geq \mathbf{I_E}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{I_E}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{I_E}(\mathbf{x}_1) \wedge \mathbf{I_E}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{I}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(ß_2).$$

For the falsity-membership function,

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \mathbf{F}_{[ß_1, ß_2] = \aleph(z)}\{\mathbf{F_E}(z)\}.$$

Hence,

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \leq \mathbf{F_E}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{F_E}([\mathbf{x}_1, \mathbf{x}_2]) \leq \mathbf{F_E}(\mathbf{x}_1) \wedge \mathbf{F_E}(\mathbf{x}_2).$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{F}_{\aleph(\mathbf{E})}(ß_2).$$

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Corollary 3.3. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T_E}, \mathbf{I_E}, \mathbf{F_E})$ is a neutrosophic Lie ideal of Ξ_1, then the image $\aleph(\mathbf{E})$ is a neutrosophic Lie ideal of* $\text{im}(\aleph)$.

Proof. Let $ß_1, ß_2 \in \mathrm{im}(\aleph)$ and let $\alpha \in \mathbb{F}$. Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that $\aleph(\mathbf{x}_1) = ß_1$ and $\aleph(\mathbf{x}_2) = ß_2$.

Now, we verify the defining conditions of a neutrosophic Lie ideal.

(i) Closure under additions:

Since $\aleph$ is linear,

$$ß_1 + ß_2 = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2) = \aleph(\mathbf{x}_1 + \mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß_1 + ß_2) = \sup_{ß_1+ß_2=\aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2).$$

Because $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(ß_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(ß_1 + ß_2) = \sup_{ß_1+ß_2=\aleph(z)} \{\mathbf{I}_{\mathbf{E}}(z)\} \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2)$$
$$\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{E}}(\mathbf{x}_2),$$

which gives

$$\mathbf{I}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(ß_2).$$

Also,

$$\mathbf{F}_{\aleph(\mathbf{E})}(ß_1 + ß_2) = \in \mathbf{F}_{ß_1+ß_2=\aleph(z)}\{\mathbf{F}_{\mathbf{E}}(z)\} \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2)$$
$$\leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{E}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E})}(ß_1 + ß_2) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(ß_2).$$

(ii) Scalar multiplication: Since

$$\alpha ß_1 = \alpha\aleph(\mathbf{x}_1) = \aleph(\alpha\mathbf{x}_1),$$

we have

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha ß_1) = \sup_{\alpha ß_1=\aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha ß_1) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1).$$

Similarly,

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and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\alpha ß_1) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1).$$

(iii) Lie bracket condition: Since $\aleph$ is a Lie algebra homomorphism,

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Therefore,

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Because $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{T}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{T}_{\aleph(\mathbf{E})}(ß_2).$$

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Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \sup_{[ß_1,ß_2]=\aleph(z)} \{\mathbf{I}_{\mathbf{E}}(z)\} \geq \mathbf{I}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2])$$

$$\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{I}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(ß_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(ß_2).$$

Also,

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) = \inf_{[ß_1,ß_2]=\aleph(z)} \{\mathbf{F}_{\mathbf{E}}(z)\} \leq \mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2])$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{E}}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{F}_{\aleph(\mathbf{E})}([ß_1, ß_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(ß_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(ß_2).$$

Consequently, $\aleph(\mathbf{E})$ satisfies all the defining conditions of a neutrosophic Lie ideal of $\mathrm{im}(\aleph)$. □

Theorem 3.3. *Let $\aleph : \Xi_1 \to \Xi_2$ be an injective Lie algebra homomorphism and let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic Lie subalgebra of Ξ_1. Then $\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}$.*

Proof. Let $\mathbf{E}^* = \aleph^{-1}(\aleph(\mathbf{E}))$. We show that the truth-membership, indeterminacy-membership, and falsity-membership functions of $\mathbf{E}^*$ coincide with those of $\mathbf{E}$.

Let $x \in \Xi_1$ be arbitrary. Since $\aleph$ is injective,

$$\aleph^{-1}(\{\aleph(\mathbf{x})\}) = \{\mathbf{x}\}.$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \sup_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{T}_{\mathbf{E}}(\mathbf{y}) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{T}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{I}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Also,

$$\mathbf{F}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \inf_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{F}_{\mathbf{E}}(\mathbf{y}) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}),$$

and therefore,

$$\mathbf{F}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Thus,

$$\mathbf{T}_{\mathbf{E}^*} = \mathbf{T}_{\mathbf{E}}, \quad \mathbf{I}_{\mathbf{E}^*} = \mathbf{I}_{\mathbf{E}}, \quad \mathbf{F}_{\mathbf{E}^*} = \mathbf{F}_{\mathbf{E}}.$$

Hence,

$$\mathbf{E}^* = \mathbf{E}.$$

Therefore,

$$\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

□

Theorem 3.4. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism and let $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ be a neutrosophic Lie subalgebra of Ξ_2. Then $\aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P}$.*

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Proof. Let $\mathsf{P}^* = \aleph(\aleph^{-1}(\mathsf{P}))$. Let $ß \in \Xi_2$. Since $\aleph$ is surjective,

$$\aleph^{-1}(\{ß\}) \neq \varnothing.$$

For every $x \in \aleph^{-1}(\{ß\})$, we have

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})) = \mathbf{T}_{\mathsf{P}}(ß).$$

Consequently,

$$\mathbf{T}_{\mathsf{P}^*}(ß) = \sup_{x \in \aleph^{-1}(\{ß\})} \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T}_{\mathsf{P}}(ß).$$

Similarly,

$$\mathbf{I}_{\mathsf{P}^*}(ß) = \sup_{x \in \aleph^{-1}(\{ß\})} \mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{I}_{\mathsf{P}}(ß).$$

Also,

$$\mathbf{F}_{\mathsf{P}^*}(ß) = \inf_{x \in \aleph^{-1}(\{ß\})} \mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{F}_{\mathsf{P}}(ß).$$

Therefore,

$$\mathbf{T}_{\mathsf{P}^*} = \mathbf{T}_{\mathsf{P}}, \quad \mathbf{I}_{\mathsf{P}^*} = \mathbf{I}_{\mathsf{P}}, \quad \mathbf{F}_{\mathsf{P}^*} = \mathbf{F}_{\mathsf{P}}.$$

Hence,

$$\mathsf{P}^* = \mathsf{P}.$$

Thus,

$$\aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P}.$$

□

Theorem 3.5. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism and let $\mathbf{E}$ be a neutrosophic subset of Ξ_1. Then $\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E}))$.*

Proof. Let $\mathbf{E}^* = \aleph^{-1}(\aleph(\mathbf{E}))$. Take any $x \in \Xi_1$. Since

$$\mathbf{x} \in \aleph^{-1}(\{\aleph(\mathbf{x})\}),$$

it follows that

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \leq \sup_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{T}_{\mathbf{E}}(\mathbf{y}) = \mathbf{T}_{\mathbf{E}^*}(\mathbf{x}).$$

Therefore,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{T}_{\mathbf{E}^*}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}^*}(\mathbf{x}) \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Furthermore,

$$\mathbf{F}_{\aleph^{-1}(\aleph(\mathbf{E}))}(\aleph(\mathbf{x})) = \inf_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{F}_{\mathbf{E}}(\mathbf{y}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}),$$

which implies

$$\mathbf{F}_{\mathbf{E}^*}(\mathbf{x}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\mathbf{E}} \leq \mathbf{T}_{\mathbf{E}^*}, \quad \mathbf{I}_{\mathbf{E}} \leq \mathbf{I}_{\mathbf{E}^*}, \quad \mathbf{F}_{\mathbf{E}} \geq \mathbf{F}_{\mathbf{E}^*}.$$

Therefore,

$$\mathbf{E} \subseteq \mathbf{E}^*.$$

Thus,

$$\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E})).$$

□

Theorem 3.6. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism and let P be a neutrosophic subset of Ξ_2. Then $\aleph(\aleph^{-1}(\mathsf{P})) \subseteq \mathsf{P}$.*

NEUTROSOPHIC LIE SUBALGEBRAS AND IDEALS: IMAGES, INVERSE IMAGES, AND QUOTIENT STRUCTURES

MUKHTAR AHMAD[1)] AND FLORENTIN SMARANDACHE[2,*)]

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Proof. Let $\mathsf{P}^* = \aleph(\aleph^{-1}(\mathsf{P}))$. Take any $ß \in \mathrm{im}(\aleph)$. Now, we choose $x \in \Xi_1$ such that $\aleph(\mathbf{x}) = ß$. For every $\mathbf{y} \in \aleph^{-1}(\{ß\})$, we have

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{y}) = \mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{y})) = \mathbf{T}_{\mathsf{P}}(ß).$$

Hence,

$$\mathbf{T}_{\mathsf{P}^*}(ß) = \sup_{\mathbf{y} \in \aleph^{-1}(\{ß\})} \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{y}) = \mathbf{T}_{\mathsf{P}}(ß).$$

Similarly,

$$\mathbf{I}_{\mathsf{P}^*}(ß) = \mathbf{I}_{\mathsf{P}}(ß),$$

and

$$\mathbf{F}_{\mathsf{P}^*}(ß) = \mathbf{F}_{\mathsf{P}}(ß).$$

Therefore,

$$\mathsf{P}^* = \mathsf{P}\big|_{\mathrm{im}(\aleph)}.$$

Consequently,

$$\aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P}^* \subseteq \mathsf{P}.$$

Hence,

$$\aleph(\aleph^{-1}(\mathsf{P})) \subseteq \mathsf{P}.$$

□

Theorem 3.7. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra isomorphism. Then there is a one-to-one correspondence between the set of all neutrosophic Lie subalgebras of Ξ_1 and the set of all neutrosophic Lie subalgebras of Ξ_2 given by $\mathbf{E} \mapsto \aleph(\mathbf{E})$.*

Proof. Define

$$\Phi : \mathcal{NS}(\Xi_1) \to \mathcal{NS}(\Xi_2)$$

by

$$\Phi(\mathbf{E}) = \aleph(\mathbf{E}),$$

where $\mathcal{NS}(\Xi_i)$ denotes the collection of all neutrosophic Lie subalgebras of Ξ_i.

Since $\aleph$ is a Lie algebra homomorphism, the image of every neutrosophic Lie subalgebra of Ξ_1 is a neutrosophic Lie subalgebra of Ξ_2. Hence Φ is well-defined.

We first show that Φ is injective. Suppose

$$\Phi(\mathbf{E}_1) = \Phi(\mathbf{E}_2).$$

Then

$$\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2).$$

Applying $\aleph^{-1}$ to both sides yields

$$\aleph^{-1}(\aleph(\mathbf{E}_1)) = \aleph^{-1}(\aleph(\mathbf{E}_2)).$$

Since $\aleph$ is injective,

$$\aleph^{-1}(\aleph(\mathbf{E}_i)) = \mathbf{E}_i \qquad (i = 1, 2).$$

Therefore

$$\mathbf{E}_1 = \mathbf{E}_2.$$

Hence Φ is injective.

Next we show that Φ is surjective. Let P be any neutrosophic Lie subalgebra of Ξ_2. Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph^{-1}(\mathsf{P})$$

is a neutrosophic Lie subalgebra of Ξ_1.

Because $\aleph$ is surjective,

$$\aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P}.$$

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Thus

$$\Phi\big(\aleph^{-1}(\mathsf{P})\big) = \mathsf{P}.$$

Hence Φ is surjective.

Therefore Φ is bijective, establishing a one-to-one correspondence between neutrosophic Lie subalgebras of Ξ_1 and neutrosophic Lie subalgebras of Ξ_2. □

Corollary 3.4. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra isomorphism. Then there is a one-to-one correspondence between the neutrosophic Lie ideals of Ξ_1 and the neutrosophic Lie ideals of Ξ_2.*

Theorem 3.8. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism and let $\{\mathsf{P}_\lambda\}_{\lambda\in\Lambda}$ be a family of neutrosophic Lie subalgebras of Ξ_2. Then $\aleph^{-1}\Big(\bigcap_{\lambda\in\Lambda}\mathsf{P}_\lambda\Big) = \bigcap_{\lambda\in\Lambda}\aleph^{-1}(\mathsf{P}_\lambda)$.*

Proof. Let $\mathsf{P} = \bigcap_{\lambda\in\Lambda}\mathsf{P}_\lambda$. For every $x \in \Xi_1$

$$\mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})) = \inf_{\lambda\in\Lambda}\mathbf{T}_{\mathsf{P}_\lambda}(\aleph(\mathbf{x})).$$

Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \inf_{\lambda\in\Lambda}\mathbf{T}_{\aleph^{-1}(\mathsf{P}_\lambda)}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \inf_{\lambda\in\Lambda}\mathbf{I}_{\aleph^{-1}(\mathsf{P}_\lambda)}(\mathbf{x}),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \sup_{\lambda\in\Lambda}\mathbf{F}_{\aleph^{-1}(\mathsf{P}_\lambda)}(\mathbf{x}).$$

These are precisely the membership functions of $\bigcap_{\lambda\in\Lambda}\aleph^{-1}(\mathsf{P}_\lambda)$.

Hence,

$$\aleph^{-1}\Big(\bigcap_{\lambda\in\Lambda}\mathsf{P}_\lambda\Big) = \bigcap_{\lambda\in\Lambda}\aleph^{-1}(\mathsf{P}_\lambda).$$

□

Theorem 3.9. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism. If $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ_1 such that $\ker(\aleph) \subset \mathbf{E}$, then $\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}$.*

Proof. In view of Theorem 3.5, we have

$$\mathbf{E} \subset \aleph^{-1}(\aleph(\mathbf{E})).$$

It remains to prove the reverse inclusion. Let $\mathbf{x} \in \aleph^{-1}(\aleph(\mathbf{E}))$. Then, there exists $e \in \mathbf{E}$ such that

$$\aleph(\mathbf{x}) = \aleph(e).$$

Hence,

$$\aleph(\mathbf{x} - e) = 0,$$

so that

$$\mathbf{x} - e \in \ker(\aleph).$$

Since

$$\ker(\aleph) \subseteq \mathbf{E},$$

we have

$$\mathbf{x} - e \in \mathbf{E}.$$

Because $\mathbf{E}$ is a neutrosophic Lie ideal and therefore a neutrosophic Lie subalgebra, it is closed under addition. Thus,

$$\mathbf{x} = (\mathbf{x} - e) + e \in \mathbf{E}.$$

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Hence,

$$\aleph^{-1}(\aleph(\mathbf{E})) \subseteq \mathbf{E}.$$

Combining this with

$$\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E}))$$

we obtain

$$\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

□

Theorem 3.10. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism. Then, the mapping $\mathbf{E} \longmapsto \aleph(\mathbf{E})$ establishes a bijection between*

(1) *the neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$, and*

(2) *the neutrosophic Lie ideals of Ξ_2.*

The inverse correspondence is given by $\mathsf{P} \longmapsto \aleph^{-1}(\mathsf{P})$.

Proof. Let

$$\mathcal{A} = \{\mathbf{E} : \mathbf{E} \text{ is a neutrosophic Lie ideal of } \Xi_1, \ker(\aleph) \subseteq \mathbf{E}\}$$

and

$$\mathcal{B} = \{\mathsf{P} : \mathsf{P} \text{ is a neutrosophic Lie ideal of } \Xi_2\}.$$

Define

$$\Phi : \mathcal{A} \to \mathcal{B}, \qquad \Phi(\mathbf{E}) = \aleph(\mathbf{E}).$$

Since the image of a neutrosophic Lie ideal under a Lie homomorphism is again a neutrosophic Lie ideal, Φ is well-defined.

Define

$$\Psi : \mathcal{B} \to \mathcal{A}, \qquad \Psi(\mathsf{P}) = \aleph^{-1}(\mathsf{P}).$$

Since inverse images of neutrosophic Lie ideals are neutrosophic Lie ideals, Ψ is well-defined.

Moreover, $\ker(\aleph) = \aleph^{-1}(\{0\}) \subseteq \aleph^{-1}(\mathsf{P})$, because every ideal contains the zero element. Thus, $\Psi(\mathsf{P}) \in \mathcal{A}$. Now, let $\mathbf{E} \in \mathcal{A}$. By Theorem 3.9,

$$\Psi(\Phi(\mathbf{E})) = \aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

Likewise, for every $\mathsf{P} \in \mathcal{B}$,

$$\Phi(\Psi(\mathsf{P})) = \aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P},$$

since $\aleph$ is surjective. Therefore,

$$\Psi \circ \Phi = \mathrm{id}_{\mathcal{A}}$$

and

$$\Phi \circ \Psi = \mathrm{id}_{\mathcal{B}}.$$

Hence, Φ and Ψ are inverse bijections.

Consequently, the correspondence $\mathbf{E} \leftrightarrow \aleph(\mathbf{E})$ is a bijection between neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and neutrosophic Lie ideals of Ξ_2. □

4. More Homomorphism Properties of Neutrosophic Lie Algebras

Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ be two neutrosophic sets on a Lie algebra Ξ over a field $\mathbb{F}$. The sum of $\mathbf{E}$ and P is defined by $\mathbf{E} + \mathsf{P} = (\mathbf{T}_{\mathbf{E}+\mathsf{P}}, \mathbf{I}_{\mathbf{E}+\mathsf{P}}, \mathbf{F}_{\mathbf{E}+\mathsf{P}})$, where for every $\mathbf{x} \in \Xi$,

$$\mathbf{T}_{\mathbf{E}+\mathsf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2)\},$$

$$\mathbf{I}_{\mathbf{E}+\mathsf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathsf{P}}(\mathbf{x}_2)\},$$

and

$$\mathbf{F}_{\mathbf{E}+\mathsf{P}}(\mathbf{x}) = \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathsf{P}}(\mathbf{x}_2)\}.$$

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Now, let

$$\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}}) \quad \text{and} \quad Q = (\mathbf{T}_Q, \mathbf{I}_Q, \mathbf{F}_Q)$$

be two neutrosophic sets on the same set Ξ.

We say that $\mathbf{E}$ is homogeneous with Q if, for all $\mathbf{x}, \beta \in \Xi$, the following conditions hold:

(1) $\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{T}_Q(\beta)$;
(2) $\mathbf{I}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{I}_Q(\beta)$;
(3) $\mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{F}_Q(\beta)$.

Theorem 4.1. *Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ be two neutrosophic Lie ideals of a Lie algebra Ξ over a field $\mathbb{F}$. Assume that $\mathbf{E}$ is homogeneous with P and that the sum $\mathbf{E}+\mathsf{P}$ is homogeneous. Then, $\mathbf{E} + \mathsf{P}$ is a neutrosophic Lie ideal of Ξ.*

Proof. Let

$$\widehat{\mathsf{A}} = \mathbf{E} + \mathsf{P} = (\mathbf{T}_{\mathbf{E}+\mathsf{P}}, \mathbf{I}_{\mathbf{E}+\mathsf{P}}, \mathbf{F}_{\mathbf{E}+\mathsf{P}}).$$

Now, we show that $\widehat{\mathsf{A}}$ satisfies all the defining conditions of a neutrosophic Lie ideal. Let $\mathbf{x}, \mathbb{b} \in \Xi$ and $\alpha \in \mathbb{F}$. Then, we have

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2)\},$$

$$\mathbf{I}_{\widehat{\mathsf{A}}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathsf{P}}(\mathbf{x}_2)\},$$

and

$$\mathbf{F}_{\widehat{\mathsf{A}}}(\mathbf{x}) = \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathsf{P}}(\mathbf{x}_2)\}.$$

(i) Closure under addition. Take decompositions

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad \mathbb{b} = \mathbb{b}_1 + \mathbb{b}_2,$$

where

$$\mathbf{x}_1, \mathbb{b}_1 \in \Xi \quad \text{and} \quad \mathbf{x}_2, \mathbb{b}_2 \in \Xi.$$

Then

$$\mathbf{x} + \mathbb{b} = (\mathbf{x}_1 + \mathbb{b}_1) + (\mathbf{x}_2 + \mathbb{b}_2).$$

Hence,

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) = \sup_{\mathbf{x}+\mathbb{b}=u+v} \{\mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathsf{P}}(v)\}$$
$$\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbb{b}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2 + \mathbb{b}_2).$$

Since $\mathbf{E}$ and P are neutrosophic Lie ideals,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbb{b}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbb{b}_1),$$

and

$$\mathbf{T}_{\mathsf{P}}(\mathbf{x}_2 + \mathbb{b}_2) \geq \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2) \wedge \mathbf{T}_{\mathsf{P}}(\mathbb{b}_2).$$

Therefore,

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) \geq (\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbb{b}_1)) \wedge (\mathbf{T}_{\mathsf{P}}(\mathbf{x}_2) \wedge \mathbf{T}_{\mathsf{P}}(\mathbb{b}_2)).$$

Using associativity and commutativity of $\wedge$, we rearrange

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) \geq (\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2)) \wedge (\mathbf{T}_{\mathbf{E}}(\mathbb{b}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbb{b}_2)).$$

Taking supremum over all decompositions, we obtain

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) \geq \mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x}) \wedge \mathbf{T}_{\widehat{\mathsf{A}}}(\mathbb{b}).$$

Similarly,

$$\mathbf{I}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) \geq \mathbf{I}_{\widehat{\mathsf{A}}}(\mathbf{x}) \wedge \mathbf{I}_{\widehat{\mathsf{A}}}(\mathbb{b}).$$

For the falsity-membership function,

$$\mathbf{F}_{\widehat{\mathsf{A}}}(\mathbf{x} + \mathbb{b}) = \in \mathbf{F}_{\mathbf{x}+\mathbb{b}=u+v} \{\mathbf{F}_{\mathbf{E}}(u) \vee \mathbf{F}_{\mathsf{P}}(v)\}.$$

NEUTROSOPHIC LIE SUBALGEBRAS AND IDEALS: IMAGES, INVERSE IMAGES, AND QUOTIENT STRUCTURES

MUKHTAR AHMAD[1)] AND FLORENTIN SMARANDACHE[2,*)]

[1)]Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung 40132, Indonesia

[2)]University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

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1. Introduction

The theory of fuzzy sets, introduced by Zadeh [1], is one of the most active and influential areas of modern mathematics due to its wide range of applications in algebra, engineering, artificial intelligence, information sciences, and decision-making problems. The pioneering work of [1] introduced the notion of fuzzy sets, which provided a mathematical framework for dealing with vagueness and imprecise information. Since then, fuzzy set theory has been extensively generalized and applied in many branches of mathematics and theoretical sciences.

To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

Since

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1 + ß_1) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{E}}(ß_1),$$

and

$$\mathbf{F}_{\mathbf{P}}(\mathbf{x}_2 + ß_2) \leq \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2) \vee \mathbf{F}_{\mathbf{P}}(ß_2),$$

we obtain

$$\mathbf{F}_{\mathsf{A}}(\mathbf{x} + ß) \leq \mathbf{F}_{\mathsf{A}}(\mathbf{x}) \vee \mathbf{F}_{\mathsf{A}}(ß).$$

(ii) Scalar multiplication: Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. Then, $\alpha\mathbf{x} = \alpha\mathbf{x}_1 + \alpha\mathbf{x}_2$. Hence,

$$\mathbf{T}_{\mathsf{A}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \wedge \mathbf{T}_{\mathsf{P}}(\alpha\mathbf{x}_2).$$

Since $\mathbf{E}$ and P are neutrosophic Lie ideals,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1)$$

and

$$\mathbf{T}_{\mathsf{P}}(\alpha\mathbf{x}_2) \geq \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{T}_{\mathsf{A}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2).$$

Taking supremum over all decompositions,

$$\mathbf{T}_{\widehat{\mathsf{A}}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\widehat{\mathsf{A}}}(\alpha\mathbf{x}) \geq \mathbf{I}_{\widehat{\mathsf{A}}}(\mathbf{x}),$$

and

$$\mathbf{F}_{\widehat{\mathsf{A}}}(\alpha\mathbf{x}) \geq \mathbf{F}_{\widehat{\mathsf{A}}}(\mathbf{x}).$$

(iii) Lie bracket condition: Take decompositions

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad ß = ß_1 + ß_2.$$

Using bilinearity of the Lie bracket,

$$[\mathbf{x}, ß] = [\mathbf{x}_1 + \mathbf{x}_2, ß_1 + ß_2].$$

Hence,

$$[\mathbf{x}, ß] = [\mathbf{x}_1, ß_1] + [\mathbf{x}_1, ß_2] + [\mathbf{x}_2, ß_1] + [\mathbf{x}_2, ß_2].$$

Since $\mathbf{E}$ and P are neutrosophic Lie ideals,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, ß_1]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{T}_{\mathbf{E}}(ß_1),$$

$$\mathbf{T}_{\mathsf{P}}([\mathbf{x}_2, ß_2]) \geq \mathbf{T}_{\mathsf{P}}(\mathbf{x}_2) \vee \mathbf{T}_{\mathsf{P}}(ß_2).$$

Therefore,

$$\mathbf{T}_{\widehat{\mathsf{A}}}([\mathbf{x}, ß]) \geq \mathbf{T}_{\widehat{\mathsf{A}}}(\mathbf{x}) \vee \mathbf{T}_{\widehat{\mathsf{A}}}(ß).$$

Similarly,

$$\mathbf{I}_{\widehat{\mathsf{A}}}([\mathbf{x}, ß]) \geq \mathbf{I}_{\widehat{\mathsf{A}}}(\mathbf{x}) \wedge \mathbf{I}_{\widehat{\mathsf{A}}}(ß)$$

and

$$\mathbf{F}_{\widehat{\mathsf{A}}}([\mathbf{x}, ß]) \leq \mathbf{F}_{\widehat{\mathsf{A}}}(\mathbf{x}) \wedge \mathbf{F}_{\widehat{\mathsf{A}}}(ß).$$

Consequently, $\mathbf{E} + \mathsf{P}$ satisfies all the defining conditions of a neutrosophic Lie ideal of Ξ. □

Theorem 4.2. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism. Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ be neutrosophic Lie ideals of Ξ_1 such that $\mathbf{E}$ is homogeneous with respect to P. Then, $\aleph(\mathbf{E} + \mathsf{P}) = \aleph(\mathbf{E}) + \aleph(\mathsf{P})$.*

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Proof. Let $ß \in \Xi_2$. We prove the equality componentwise.

(i) Truth-membership function: By definition, we have

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=\aleph(\mathbf{x})} \{\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Since

$$\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2)\},$$

we obtain

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=\aleph(\mathbf{x})} \left\{ \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} (\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2)) \right\}.$$

Because $\aleph$ is linear and surjective,

$$ß = \aleph(\mathbf{x}) = \aleph(\mathbf{x}_1 + \mathbf{x}_2) = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2).$$

Now, let

$$u = \aleph(\mathbf{x}_1) \quad \text{and} \quad v = \aleph(\mathbf{x}_2).$$

Then,

$$ß = u + v.$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=u+v} \left\{ \sup_{u=\aleph(\mathbf{x}_1)} \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \sup_{v=\aleph(\mathbf{x}_2)} \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2) \right\}.$$

By definition of image anti-neutrosophic sets,

$$\sup_{u=\aleph(\mathbf{x}_1)} \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) = \mathbf{T}_{\aleph(\mathbf{E})}(u),$$

and

$$\sup_{v=\aleph(\mathbf{x}_2)} \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2) = \mathbf{T}_{\aleph(\mathbf{P})}(v).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=u+v} \{\mathbf{T}_{\aleph(\mathbf{E})}(u) \wedge \mathbf{T}_{\aleph(\mathbf{P})}(v)\}.$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \mathbf{T}_{\aleph(\mathbf{E})+\aleph(\mathbf{P})}(ß).$$

(ii) Indeterminacy-membership function: Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=\aleph(\mathbf{x})} \{\mathbf{I}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Using the definition of the sum,

$$\mathbf{I}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{P}}(\mathbf{x}_2)\}.$$

Proceeding, as above, we obtain

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \sup_{ß=u+v} \{\mathbf{I}_{\aleph(\mathbf{E})}(u) \wedge \mathbf{I}_{\aleph(\mathbf{P})}(v)\}.$$

Thus,

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \mathbf{I}_{\aleph(\mathbf{E})+\aleph(\mathbf{P})}(ß).$$

(iii) Falsity-membership function: By definition,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(ß) = \in \mathbf{F}_{ß=\aleph(\mathbf{x})}\{\mathbf{F}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Since

$$\mathbf{F}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2}\{\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2)\},$$

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we have

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathsf{P})}(ß) = \in \mathbf{F}_{ß=\aleph(\mathbf{x})}\left\{\in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2}\left(\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathsf{P}}(\mathbf{x}_2)\right)\right\}.$$

Using surjectivity and linearity of $\aleph$,

$$ß = u + v,$$

where

$$u = \aleph(\mathbf{x}_1) \quad \text{and} \quad v = \aleph(\mathbf{x}_2).$$

Hence,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathsf{P})}(ß) = \in \mathbf{F}_{ß=u+v}\left\{\in \mathbf{F}_{u=\aleph(\mathbf{x}_1)}\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \in \mathbf{F}_{v=\aleph(\mathbf{x}_2)}\mathbf{F}_{\mathsf{P}}(\mathbf{x}_2)\right\}.$$

By definition,

$$\in \mathbf{F}_{u=\aleph(\mathbf{x}_1)}\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) = \mathbf{F}_{\aleph(\mathbf{E})}(u),$$

and

$$\in \mathbf{F}_{v=\aleph(\mathbf{x}_2)}\mathbf{F}_{\mathsf{P}}(\mathbf{x}_2) = \mathbf{F}_{\aleph(\mathsf{P})}(v).$$

Therefore,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathsf{P})}(ß) = \in \mathbf{F}_{ß=u+v}\left\{\mathbf{F}_{\aleph(\mathbf{E})}(u) \vee \mathbf{F}_{\aleph(\mathsf{P})}(v)\right\}.$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathsf{P})}(ß) = \mathbf{F}_{\aleph(\mathbf{E})+\aleph(\mathsf{P})}(ß).$$

Consequently,

$$\aleph(\mathbf{E}+\mathsf{P}) = \aleph(\mathbf{E}) + \aleph(\mathsf{P}).$$

□

Theorem 4.3. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra isomorphism. Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathsf{P} = (\mathbf{T}_{\mathsf{P}}, \mathbf{I}_{\mathsf{P}}, \mathbf{F}_{\mathsf{P}})$ be neutrosophic Lie ideals of Ξ_1. If $\mathbf{E}$ is homogeneous with P, then $\aleph(\mathbf{E})$ is homogeneous with $\aleph(\mathsf{P})$.*

Proof. Since $\aleph$ is a Lie algebra isomorphism, it is in particular bijective. Let $ß, \tau \in \Xi_2$ be arbitrary. Since $\aleph$ is surjective, there exist $\mathbf{x}, y \in \Xi_1$ such that

$$\aleph(\mathbf{x}) = ß \quad \text{and} \quad \aleph(y) = \tau.$$

Because $\aleph$ is injective,

$$\aleph^{-1}(ß) = \{x\} \quad \text{and} \quad \aleph^{-1}(\tau) = \{y\}.$$

Hence, by the definition of the image of a neutrosophic set,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß) = \sup_{u \in \aleph^{-1}(ß)} \mathbf{T}_{\mathbf{E}}(u) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}),$$

and

$$\mathbf{T}_{\aleph(\mathsf{P})}(\tau) = \sup_{v \in \aleph^{-1}(\tau)} \mathbf{T}_{\mathsf{P}}(v) = \mathbf{T}_{\mathsf{P}}(y).$$

Since $\mathbf{E}$ is homogeneous with P, we have

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{T}_{\mathsf{P}}(y).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß) \leq \mathbf{T}_{\aleph(\mathsf{P})}(\tau).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(ß) = \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{I}_{\mathsf{P}}(y) = \mathbf{I}_{\aleph(\mathsf{P})}(\tau),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(ß) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{F}_{\mathsf{P}}(y) = \mathbf{F}_{\aleph(\mathsf{P})}(\tau).$$

Thus, for all $ß, \tau \in \Xi_2$,

$$\mathbf{T}_{\aleph(\mathbf{E})}(ß) \leq \mathbf{T}_{\aleph(\mathsf{P})}(\tau),$$
$$\mathbf{I}_{\aleph(\mathbf{E})}(ß) \leq \mathbf{I}_{\aleph(\mathsf{P})}(\tau),$$

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and

$$\mathbf{F}_{\aleph(\mathbf{E})}(ß) \leq \mathbf{F}_{\aleph(\mathsf{P})}(\tau).$$

Hence, $\aleph(\mathbf{F})$ is homogeneous with $\aleph(\mathsf{P})$. □

Remark 4.1. *The assumption that $\aleph$ is a Lie algebra isomorphism is stronger than necessary. In fact, the proof only uses the surjectivity of $\aleph$ to guarantee the existence of preimages of elements of Ξ_2. Therefore, the conclusion remains valid if $\aleph : \Xi_1 \to \Xi_2$ is merely a surjective Lie algebra homomorphism. Consequently, homogeneity of neutrosophic Lie ideals is preserved under surjective Lie algebra homomorphisms.*

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Thus,

$$\mathbf{x} \in \ker(\aleph),$$

and consequently

$$\mathbf{x} + \ker(\aleph) = \ker(\aleph)$$

which is the zero element of $\Xi_1/\ker(\aleph)$. Therefore

$$\Phi(\mathbf{x} + \ker(\aleph)) = \aleph(\mathbf{x}) = 0_{\Xi_2},$$

and Φ is injective.

To prove surjectivity, let $y \in \mathrm{im}(\aleph)$. Then there exists $\mathbf{x} \in \Xi_1$ such that $y = \aleph(\mathbf{x})$. Therefore, $y = \Phi(\mathbf{x} + \ker(\aleph))$.

Hence, Φ is surjective.

Thus, Φ is a Lie algebra isomorphism and

$$\Xi_1/\ker(\aleph) \cong \mathrm{im}(\aleph).$$

For the second assertion, consider the surjective Lie algebra homomorphism

$$\pi : \Xi_1 \to \mathrm{im}(\aleph), \quad \pi(\mathbf{x}) = \aleph(\mathbf{x}).$$

Its kernel is

$$\ker(\pi) = \ker(\aleph).$$

By the Correspondence Theorem for Homogeneous Neutrosophic Lie Ideals, the assignment $\mathbf{E} \longmapsto \pi(\mathbf{E})$ induces a bijection between the homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and the homogeneous neutrosophic Lie ideals of $\mathrm{im}(\aleph)$.

The inverse correspondence is given by

$$\mathsf{P} \mapsto \pi^{-1}(\mathsf{P}).$$

Therefore, the homogeneous neutrosophic Lie ideals of $\mathrm{im}(\aleph)$ correspond bijectively to the homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$. □

Corollary 4.1. *Let $\aleph : \Xi_1 \to \Xi_2$ be a fuzzy set, [2] introduced intuitionistic fuzzy sets by incorporating the degree of hesitancy, thereby providing a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.*

Proof. This corollary is often viewed as the most meaningful neutrosophic consequence of the First Isomorphism Theorem. □

Theorem 4.5. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism and $\mathbf{E}$ and P be neutrosophic subsets of Ξ_2. Then,*

$$\aleph^{-1}(\mathbf{E} + \mathsf{P}) = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

Proof. Let $\mathbf{R} = \aleph^{-1}(\mathbf{E} + \mathsf{P})$ and $\mathbf{S} = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P})$.

We shall prove that $\mathbf{S} \subseteq \mathbf{R}$. Let $\mathsf{x} \in \Xi_1$ be arbitrary.

By definition of inverse image, we have

$$\mathbf{T}_{\mathbf{R}}(\mathsf{x}) = \mathbf{T}_{\mathbf{E}+\mathsf{P}}(\aleph(\mathsf{x})).$$

Since

$$\mathbf{T}_{\mathbf{E}+\mathsf{P}}(\aleph(\mathsf{x})) = \sup_{\aleph(\mathsf{x})=u+v} \{\mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathsf{P}}(v)\},$$

we have

$$\mathbf{T}_{\mathbf{R}}(\mathsf{x}) = \sup_{\aleph(\mathsf{x})=u+v} \Big\{\mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathsf{P}}(v)\Big\}.$$

On the other hand,

$$\mathbf{T}_{\mathbf{S}}(\mathsf{x}) = \sup_{\mathsf{x}=\mathsf{x}_1+\mathsf{x}_2} \Big\{\mathbf{T}_{\aleph^{-1}(\mathbf{E})}(\mathsf{x}_1) \wedge \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathsf{x}_2)\Big\}.$$

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Abstract. In this paper, we study neutrosophic Lie subalgebras and neutrosophic Lie ideals under Lie algebra homomorphisms. We investigate the behavior of these structures through images and inverse images and establish several preservation properties related to Lie algebra morphisms. We also examine sums of neutrosophic Lie ideals, upper bounds, and quotient structures associated with kernels of homomorphisms. This work extends known results tothe context of neutrosophic Lie algebras, contributing new insights into their behavior under algebraic mappings.

1. Introduction

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To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating the degree of hesitancy, thereby providing a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

To overcome this limitation, [6] introduced complex fuzzy sets by extending the membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

Using the definition of inverse image, we obtain

$$\mathbf{T_S}(\mathsf{x}) = \sup_{\mathsf{x}=\mathsf{x}_1+\mathsf{x}_2} \Big\{ \mathbf{T_E}(\aleph(\mathsf{x}_1)) \wedge \mathbf{T_P}(\aleph(\mathsf{x}_2)) \Big\}.$$

Since $\aleph$ is linear,

$$\aleph(\mathsf{x}) = \aleph(\mathsf{x}_1 + \mathsf{x}_2) = \aleph(\mathsf{x}_1) + \aleph(\mathsf{x}_2).$$

Hence, every decomposition $\mathsf{x} = \mathsf{x}_1 + \mathsf{x}_2$ induces a decomposition $\aleph(\mathsf{x}) = u + v$ with $u = \aleph(\mathsf{x}_1)$ and $v = \aleph(\mathsf{x}_2)$. Therefore,

$$\mathbf{T_S}(\mathsf{x}) \leq \sup_{\aleph(\mathsf{x})=u+v} \Big\{ \mathbf{T_E}(u) \wedge \mathbf{T_P}(v) \Big\} = \mathbf{T_R}(\mathsf{x}).$$

Similarly,

$$\mathbf{I_S}(\mathsf{x}) \leq \mathbf{I_R}(\mathsf{x}).$$

For the falsity-membership function,

$$\mathbf{F_S}(\mathsf{x}) = \inf_{\mathsf{x}=\mathsf{x}_1+\mathsf{x}_2} \Big\{ \mathbf{F_E}(\aleph(\mathsf{x}_1)) \vee \mathbf{F_P}(\aleph(\mathsf{x}_2)) \Big\}.$$

Since every decomposition $\mathsf{x} = \mathsf{x}_1 + \mathsf{x}_2$ gives a decomposition $\aleph(\mathsf{x}) = \aleph(\mathsf{x}_1) + \aleph(\mathsf{x}_2)$, the set over which the above infimum is taken is contained in the set used in the definition of $\mathbf{F_R}(\mathsf{x})$. Consequently,

$$\mathbf{F_S}(\mathsf{x}) \geq \mathbf{F_R}(\mathsf{x}).$$

Thus,

$$\mathbf{T_S} \leq \mathbf{T_R}, \qquad \mathbf{I_S} \leq \mathbf{I_R}, \qquad \mathbf{F_S} \geq \mathbf{F_R}.$$

Hence,

$$\mathbf{S} \subseteq \mathbf{R}.$$

Therefore,

$$\aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}) \subseteq \aleph^{-1}(\mathbf{E} + \mathsf{P}),$$

or equivalently,

$$\aleph^{-1}(\mathbf{E} + \mathsf{P}) \supseteq \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

□

Theorem 4.6. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism. For any neutrosophic subsets $\mathbf{E}$ and P of Ξ_2,*

$$\aleph^{-1}(\mathbf{E} + \mathsf{P}) = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

Proof. By the Theorem 4.5, we have

$$\aleph^{-1}(\mathbf{E} + \mathsf{P}) \supseteq \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

It remains to prove the reverse inclusion. Let

$$\mathbf{R} = \aleph^{-1}(\mathbf{E} + \mathsf{P})$$

and

$$\mathbf{S} = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

By introducing the independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

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Since

$$\mathbf{T_{E+P}}(\aleph(\mathsf{x})) = \sup_{\aleph(\mathsf{x})=u+v} \Big\{ \mathbf{T_E}(u) \wedge \mathbf{T_P}(v) \Big\},$$

and $\aleph$ is surjective, for every decomposition $\aleph(\mathsf{x}) = u + v$ there exist $\mathsf{x}_1, \mathsf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathsf{x}_1) = u, \qquad \aleph(\mathsf{x}_2) = v.$$

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MUKHTAR AHMAD [1)] AND FLORENTIN SMARANDACHE [2)]

[1)]Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia

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Consequently,

$$\aleph(\mathsf{x}_1 + \mathsf{x}_2) = u + v = \aleph(\mathsf{x}).$$

Hence,

$$\mathsf{x} - (\mathsf{x}_1 + \mathsf{x}_2) \in \ker(\aleph).$$

Replacing x_1 by $\mathsf{x}_1 + k$ where $k = \mathsf{x} - (\mathsf{x}_1 + \mathsf{x}_2) \in \ker(\aleph)$, we obtain a decomposition

$$\mathsf{x} = (\mathsf{x}_1 + k) + \mathsf{x}_2$$

such that

$$\aleph(\mathsf{x}_1 + k) = u, \quad \aleph(\mathsf{x}_2) = v.$$

Therefore, every decomposition $\aleph(\mathsf{x}) = u + v$ arises from a decomposition $\mathsf{x} = y_1 + y_2$ with $\aleph(y_1) = u$ and $\aleph(y_2) = v$.

It follows that

$$\begin{aligned}\mathbf{T}_{\mathbf{R}}(\mathsf{x}) &= \sup_{\aleph(\mathsf{x})=u+v} \left\{\mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathbf{F}}(v)\right\} \\ &\leq \sup_{\mathsf{x}=y_1+y_2} \left\{\mathbf{T}_{\mathbf{E}}(\aleph(y_1)) \wedge \mathbf{T}_{\mathbf{F}}(\aleph(y_2))\right\} \\ &= \mathbf{T}_{\mathbf{S}}(\mathsf{x}).\end{aligned}$$

Thus,

$$\mathbf{T}_{\mathbf{R}}(\mathsf{x}) \leq \mathbf{T}_{\mathbf{S}}(\mathsf{x}).$$

By exactly the same argument,

$$\mathbf{I}_{\mathbf{R}}(\mathsf{x}) \leq \mathbf{I}_{\mathbf{S}}(\mathsf{x}).$$

For the falsity-membership function,

$$\begin{aligned}\mathbf{F}_{\mathbf{R}}(\mathsf{x}) &= \inf_{\aleph(\mathsf{x})=u+v} \left\{\mathbf{F}_{\mathbf{E}}(u) \vee \mathbf{F}_{\mathbf{F}}(v)\right\}, \\ &\leq \inf_{\mathsf{x}=y_1+y_2} \left\{\mathbf{F}_{\mathbf{E}}(\aleph(y_1)) \vee \mathbf{F}_{\mathbf{F}}(\aleph(y_2))\right\}, \\ &= \mathbf{F}_{\mathbf{S}}(\mathsf{x}).\end{aligned}$$

Hence,

$$\mathbf{F}_{\mathbf{R}}(\mathsf{x}) \geq \mathbf{F}_{\mathbf{S}}(\mathsf{x}).$$

This shows that

$$\mathbf{R} \subseteq \mathbf{S}.$$

Conversely, since $\mathbf{S}$ is the smallest neutrosophic set containing the inclusion with $\mathbf{R}$, we obtain

$$\mathbf{R} \supseteq \mathbf{S},$$

we obtain

$$\mathbf{R} = \mathbf{S}.$$

Therefore,

$$\aleph^{-1}(\mathbf{E} + \mathsf{P}) = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathsf{P}).$$

□

Remark 4.2. *The Theorem 4.5 shows that surjective Lie algebra homomorphisms commute with the neutrosophic sum operation under inverse images. Thus the inverse-image operator preserves additive neutrosophic structures.*

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Let X be a nonempty set and let $\mathbf{E} = (\mathbf{T}_\mathbf{E}, \mathbf{I}_\mathbf{E}, \mathbf{F}_\mathbf{E}) = \{(\mathbf{x}, \mathbf{T}_\mathbf{E}(\mathbf{x}), \mathbf{I}_\mathbf{E}(\mathbf{x}), \mathbf{F}_\mathbf{E}(\mathbf{x})) : \mathbf{x} \in X\}$ be a neutrosophic set on X. For $\alpha, \beta, \gamma \in [0,1]$, the set

$$\mathbf{E}_{(\alpha,\beta,\gamma)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) \geq \alpha,\ \mathbf{I}_\mathbf{E}(\mathbf{x}) \geq \beta,\ \mathbf{F}_\mathbf{E}(\mathbf{x}) \leq \gamma\} \tag{1}$$

is called the upper level subset of the neutrosophic set $\mathbf{E}$. The subsets

$$\mathbf{E}_{(\alpha^+,\beta,\gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) > \alpha,\ \mathbf{I}_\mathbf{E}(\mathbf{x}) \geq \beta,\ \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\},$$
$$\mathbf{E}_{(\alpha,\beta^+,\gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) \geq \alpha,\ \mathbf{I}_\mathbf{E}(\mathbf{x}) > \beta,\ \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\},$$

and

$$\mathbf{E}_{(\alpha^+,\beta^+,\gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) > \alpha,\ \mathbf{I}_\mathbf{E}(\mathbf{x}) > \beta,\ \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\}$$

are called strong upper level subsets of the neutrosophic set $\mathbf{E}$.

The following theorem extends the corresponding results for fuzzy and intuitionistic fuzzy Lie subalgebras to the setting of neutrosophic Lie subalgebras.

Theorem 4.7. *Let $\aleph : \Xi_1 \to \Xi_2$ be a Lie algebra homomorphism and let $\mathsf{P} = (\mathbf{T}_\mathsf{P}, \mathbf{I}_\mathsf{P}, \mathbf{F}_\mathsf{P})$ be a neutrosophic set on Ξ_2. For all $\alpha, \beta, \gamma \in [0,1]$, the following statements hold:*

(1) $\aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big) = \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)}$;

(2) $\aleph^{-1}\big(\mathsf{P}_{(\alpha^+,\beta,\gamma)}\big) = \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha^+,\beta,\gamma^-)}$;

(3) $\aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta^+,\gamma^-)}\big) = \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta^+,\gamma^-)}$;

(4) $\aleph^{-1}\big(\mathsf{P}_{(\alpha^+,\beta^+,\gamma^-)}\big) = \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha^+,\beta^+,\gamma^-)}$.

Proof. We prove (1). The proofs of (2), (3), and (4) follow similarly. Let $\mathbf{x} \in \aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big)$. Then, $\aleph(\mathbf{x}) \in P_{(\alpha,\beta,\gamma)}$. By equation (1), we have

$$\mathbf{T}_\mathsf{P}(\aleph(\mathbf{x})) \geq \alpha,\quad \mathbf{I}_\mathsf{P}(\aleph(\mathbf{x})) \geq \beta,$$

and

$$\mathbf{F}_\mathsf{P}(\aleph(\mathbf{x})) \leq \gamma.$$

Since

$$\aleph^{-1}(\mathsf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathsf{P})}, \mathbf{I}_{\aleph^{-1}(\mathsf{P})}, \mathbf{F}_{\aleph^{-1}(\mathsf{P})}),$$

where

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T}_\mathsf{P}(\aleph(\mathbf{x})),$$
$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{I}_\mathsf{P}(\aleph(\mathbf{x})),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{F}_\mathsf{P}(\aleph(\mathbf{x})),$$

it follows that

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \geq \alpha,$$
$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \geq \beta,$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \leq \gamma.$$

Hence,

$$\mathbf{x} \in \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)}.$$

Therefore,

$$\aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big) \subseteq \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)}.$$

Conversely, let

$$\mathbf{x} \in \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)}.$$

Then,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \geq \alpha,$$

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$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \geq \beta,$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) \leq \gamma.$$

Using the definition of inverse image neutrosophic sets,

$$\mathbf{T}_{\mathsf{P}}(\aleph(\mathbf{x})) \geq \alpha,$$

$$\mathbf{I}_{\mathsf{P}}(\aleph(\mathbf{x})) \geq \beta,$$

and

$$\mathbf{F}_{\mathsf{P}}(\aleph(\mathbf{x})) \leq \gamma.$$

Thus,

$$\aleph(\mathbf{x}) \in R_{(\alpha,\beta,\gamma)}.$$

Therefore,

$$\mathbf{x} \in \aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big).$$

Hence,

$$\big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)} \subseteq \aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big).$$

Consequently,

$$\aleph^{-1}\big(\mathsf{P}_{(\alpha,\beta,\gamma)}\big) = \big(\aleph^{-1}(\mathsf{P})\big)_{(\alpha,\beta,\gamma)}.$$

The proofs of (2), (3), and (4) are analogous. □

Theorem 4.8. *Let $\aleph : \Xi_1 \to \Xi_2$ be a surjective Lie algebra homomorphism and let $\mathbf{E} = (\mathbf{T}_\mathbf{E}, \mathbf{I}_\mathbf{E}, \mathbf{F}_\mathbf{E})$ be a neutrosophic Lie ideal of Ξ_1 containing $\ker(\aleph)$. Define a neutrosophic set $\overline{\mathbf{E}} = (\mathbf{T}_{\overline{\mathbf{E}}}, \mathbf{I}_{\overline{\mathbf{E}}}, \mathbf{F}_{\overline{\mathbf{E}}})$ on the quotient Lie algebra $\Xi_1/\ker(\aleph)$ by $\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{T}_\mathbf{E}(\mathbf{x})$, $\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{I}_\mathbf{E}(\mathbf{x})$, and $\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{F}_\mathbf{E}(\mathbf{x})$ for all $\mathbf{x} \in \Xi_1$. Then $\overline{\mathbf{E}}$ is a neutrosophic Lie ideal of $\Xi_1/\ker(\aleph)$.*

Proof. First, we show that $\overline{\mathbf{E}}$ is well-defined. Suppose $\mathbf{x} + \ker(\aleph) = \beta + \ker(\aleph)$. Then, $\mathbf{x} - \beta \in \ker(\aleph)$. Since $\ker(\aleph) \subset \mathbf{E}$, and $\mathbf{E}$ is a neutrosophic Lie ideal, the membership values of elements differing by an element of $\ker(\aleph)$ coincide. Hence,

$$\mathbf{T}_\mathbf{E}(\mathbf{x}) = \mathbf{T}_\mathbf{E}(\beta),$$

$$\mathbf{I}_\mathbf{E}(\mathbf{x}) = \mathbf{I}_\mathbf{E}(\beta),$$

and

$$\mathbf{F}_\mathbf{E}(\mathbf{x}) = \mathbf{F}_\mathbf{E}(\beta).$$

Therefore,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{T}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{I}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

$$\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{F}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)).$$

Thus, $\overline{\mathbf{E}}$ is well-defined on $\Xi_1/\ker(\aleph)$.

Next, let

$$\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph) \in \Xi_1/\ker(\aleph) \text{ and } \alpha \in \mathbb{F}.$$

For addition,

$$\mathbf{T}_{\overline{\mathbf{E}}}((\mathbf{x} + \ker(\aleph)) + (\beta + \ker(\aleph))) = \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph))$$

$$= \mathbf{T}_\mathbf{E}(\mathbf{x} + \beta).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_\mathbf{E}(\mathbf{x} + \beta) \geq \mathbf{T}_\mathbf{E}(\mathbf{x}) \wedge \mathbf{T}_\mathbf{E}(\beta).$$

Hence,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph)) \geq \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \wedge \mathbf{T}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

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Keywords: neutrosophic Lie algebra, neutrosophic Lie ideal, inverse image, quotient structure, homomorphism.

$$\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x}+ß+\ker(\aleph)) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)) \wedge \mathbf{I}_{\overline{\mathbf{E}}}(ß+\ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x}+ß+\ker(\aleph)) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)) \vee \mathbf{F}_{\overline{\mathbf{E}}}(ß+\ker(\aleph)).$$

For scalar multiplication,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\alpha\mathbf{x}+\ker(\aleph)) = \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}(\alpha\mathbf{x}+\ker(\aleph)) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}(\alpha\mathbf{x}+\ker(\aleph)) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)).$$

Finally, for the Lie bracket,

$$\mathbf{T}_{\overline{\mathbf{E}}}\big([\mathbf{x}+\ker(\aleph), ß+\ker(\aleph)]\big) = \mathbf{T}_{\mathbf{E}}([\mathbf{x}, ß]+\ker(\aleph)) = \mathbf{T}_{\mathbf{E}}([\mathbf{x}, ß]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}, ß]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{T}_{\mathbf{E}}(ß).$$

Therefore,

$$\mathbf{T}_{\overline{\mathbf{E}}}([\mathbf{x}+\ker(\aleph), ß+\ker(\aleph)]) \geq \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)) \vee \mathbf{T}_{\overline{\mathbf{E}}}(ß+\ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}\big([\mathbf{x}+\ker(\aleph), ß+\ker(\aleph)]\big) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)) \wedge \mathbf{I}_{\overline{\mathbf{E}}}(ß+\ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}\big([\mathbf{x}+\ker(\aleph), ß+\ker(\aleph)]\big) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x}+\ker(\aleph)) \wedge \mathbf{F}_{\overline{\mathbf{E}}}(ß+\ker(\aleph)).$$

Hence, $\mathbf{E}$ satisfies all the conditions of a neutrosophic Lie ideal of $\Xi_1/\ker(\aleph)$. □

Theorem 4.9. *Let $\aleph$ and Ξ_2 be as in Theorem 4.8, and let $\mathbf{E}$ be a neutrosophic Lie ideal of Ξ_2 containing $\ker(\aleph)$. Then $\mathbf{E}$ is the unique neutrosophic Lie ideal of Ξ_2 whose image under $\aleph$ is $(\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})})$, and Ξ_2 is the only neutrosophic Lie algebra having this property.*

Proof. We define the map $\Phi : \mathcal{N}(\Xi_2) \to \mathcal{N}(\Xi_1)$ by $\Phi(\mathbf{E}) = \aleph(\mathbf{E})$, where

$$\mathcal{N}(\Xi_1) = \{\mathbf{E} : \mathbf{E} \text{ is a neutrosophic Lie ideal of } \Xi_1 \text{ containing } \ker(\aleph)\},$$

and

$$\mathcal{N}(\Xi_2) = \{\mathsf{P} : \mathsf{P} \text{ is a neutrosophic Lie ideal of } \Xi_2\}.$$

We show that Φ is bijective.

1. Injectivity: Let $\mathbf{E}_1, \mathbf{E}_2 \in \mathcal{N}(\Xi_1)$ such that $\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2)$. Take $\mathbf{x} \in \Xi_2$. Since $\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2)$, we have

$$\mathbf{T}_{\aleph(\mathbf{E}_1)}(\aleph(\mathbf{x})) = \mathbf{T}_{\aleph(\mathbf{E}_2)}(\aleph(\mathbf{x})).$$

Thus,

$$\sup_{\aleph(a)=\aleph(\mathbf{x})} \mathbf{T}_{\mathbf{E}_1}(a) = \sup_{\aleph(b)=\aleph(\mathbf{x})} \mathbf{T}_{\mathbf{E}_2}(b).$$

Because

$$\ker(\aleph) \subseteq \mathbf{E}_1 \quad \text{and} \quad \ker(\aleph) \subseteq \mathbf{E}_2,$$

all elements in the same coset of $\ker(\aleph)$ have identical neutrosophic values. Hence,

$$\mathbf{T}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{T}_{\mathbf{E}_2}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{I}_{\mathbf{E}_2}(\mathbf{x}),$$

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To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

and

$$\mathbf{F}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{F}_{\mathbf{E}_2}(\mathbf{x}).$$

Therefore, $\mathbf{E}_1 = \mathbf{E}_2$. Hence, Φ is injective.

2. Surjectivity: Let $\mathsf{P} = (\mathbf{T}_\mathsf{P}, \mathbf{I}_\mathsf{P}, \mathbf{F}_\mathsf{P})$ be a neutrosophic Lie ideal of Ξ_2.

Now, we define

$$\aleph^{-1}(\mathsf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathsf{P})}, \mathbf{I}_{\aleph^{-1}(\mathsf{P})}, \mathbf{F}_{\aleph^{-1}(\mathsf{P})})$$

on Ξ_1 by

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{T}_\mathsf{P}(\aleph(\mathbf{x})),$$

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{I}_\mathsf{P}(\aleph(\mathbf{x})),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \mathbf{F}_\mathsf{P}(\aleph(\mathbf{x})).$$

By previous results, $\aleph^{-1}(\mathsf{P})$ is a neutrosophic Lie ideal of Ξ_1. Moreover, for every $k \in \ker(\aleph)$, we have

$$\aleph(k) = 0.$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathsf{P})}(k) = \mathbf{T}_\mathsf{P}(0),$$

$$\mathbf{I}_{\aleph^{-1}(\mathsf{P})}(k) = \mathbf{I}_\mathsf{P}(0),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathsf{P})}(k) = \mathbf{F}_\mathsf{P}(0).$$

Thus,

$$\ker(\aleph) \subseteq \aleph^{-1}(\mathsf{P}).$$

Now, consider $\aleph(\aleph^{-1}(\mathsf{P}))$. For any $\beta \in \Xi_2$, since $\aleph$ is surjective, there exists $x \in \Xi_1$ such that $\aleph(x) = \beta$. Then,

$$\mathbf{T}_{\aleph(\aleph^{-1}(\mathsf{P}))}(\beta) = \sup_{\aleph(\mathbf{x})=\beta} \mathbf{T}_{\aleph^{-1}(\mathsf{P})}(\mathbf{x}) = \sup_{\aleph(\mathbf{x})=\beta} \mathbf{T}_\mathsf{P}(\aleph(\mathbf{x})).$$

Since

$$\aleph(\mathbf{x}) = \beta,$$

we obtain

$$\mathbf{T}_{\aleph(\aleph^{-1}(\mathsf{P}))}(\beta) = \mathbf{T}_\mathsf{P}(\beta).$$

Similarly,

$$\mathbf{I}_{\aleph(\aleph^{-1}(\mathsf{P}))}(\beta) = \mathbf{I}_\mathsf{P}(\beta),$$

and

$$\mathbf{F}_{\aleph(\aleph^{-1}(\mathsf{P}))}(\beta) = \mathbf{F}_\mathsf{P}(\beta).$$

Therefore,

$$\aleph(\aleph^{-1}(\mathsf{P})) = \mathsf{P}.$$

Hence, Φ is surjective.

Consequently, Φ is a bijection between neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and neutrosophic Lie ideals of Ξ_2. □

Theorem 4.10. *Let* $\mathbf{E} = (\mathbf{T}_\mathbf{E}, \mathbf{I}_\mathbf{E}, \mathbf{F}_\mathbf{E})$ *be a neutrosophic set on a nonempty set* X*. If* $\alpha_1 \leq \alpha_2$*,* $\beta_1 \leq \beta_2$*, and* $\gamma_1 \leq \gamma_2$*. Then,* $\mathbf{E}_{(\alpha_2,\beta_2,\gamma_1)} \subseteq \mathbf{E}_{(\alpha_1,\beta_1,\gamma_2)}$.

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To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by assigning both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

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Proof. Let

$$\mathbf{x} \in \mathbf{E}_{(\alpha_2,\beta_2,\gamma_1)}.$$

Then, by the definition of upper level subsets,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha_2, \qquad \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \geq \beta_2, \qquad \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma_2.$$

Since

$$\alpha_1 \leq \alpha_2, \qquad \beta_1 \leq \beta_2, \qquad \gamma_1 \leq \gamma_2,$$

it follows that

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha_1, \qquad \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \geq \beta_1, \qquad \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma_2.$$

Hence

$$\mathbf{x} \in \mathbf{E}_{(\alpha_1,\beta_1,\gamma_2)}.$$

Therefore,

$$\mathbf{E}_{(\alpha_2,\beta_2,\gamma_1)} \subseteq \mathbf{E}_{(\alpha_1,\beta_1,\gamma_2)}.$$

□

Theorem 4.11. *Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic set on X. Then, for every $\mathbf{x} \in X$,*

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \sup\left\{\alpha \in [0,1] : \mathbf{x} \in \mathbf{E}_{(\alpha,0,1)}\right\},$$

$$\mathbf{I}_{\mathbf{E}}(\mathbf{x}) = \sup\left\{\beta \in [0,1] : \mathbf{x} \in \mathbf{E}_{(0,\beta,1)}\right\},$$

and

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = \inf\left\{\gamma \in [0,1] : \mathbf{x} \in \mathbf{E}_{(0,0,\gamma)}\right\}.$$

Proof. Fix $\mathbf{x} \in X$ and consider the set

$$A_{\mathbf{x}} = \{\alpha \in [0,1] : \mathbf{x} \in \mathbf{E}_{(\alpha,0,1)}\}.$$

By definition,

$$\mathbf{x} \in \mathbf{E}_{(\alpha,0,1)} \iff \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha.$$

Hence,

$$A_{\mathbf{x}} = [0, \mathbf{T}_{\mathbf{E}}(\mathbf{x})].$$

Therefore,

$$\sup A_{\mathbf{x}} = \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Thus,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \sup\{\alpha \in [0,1] : \mathbf{x} \in \mathbf{E}_{(\alpha,0,1)}\}.$$

Similarly,

$$B_{\mathbf{x}} = \{\beta \in [0,1] : \mathbf{x} \in \mathbf{E}_{(0,\beta,1)}\} = [0, \mathbf{I}_{\mathbf{E}}(\mathbf{x})].$$

Thus,

$$\sup B_{\mathbf{x}} = \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Finally,

$$C_{\mathbf{x}} = \left\{\gamma \in [0,1] : \mathbf{x} \in \mathbf{E}_{(0,0,\gamma)}\right\}.$$

Since

$$\mathbf{x} \in \mathbf{E}_{(0,0,\gamma)} \iff \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma,$$

we obtain

$$C_{\mathbf{x}} = [\mathbf{F}_{\mathbf{E}}(\mathbf{x}), 1].$$

Hence,

$$\inf C_{\mathbf{x}} = \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Therefore,

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = \inf\left\{\gamma \in [0,1] : \mathbf{x} \in \mathbf{E}_{(0,0,\gamma)}\right\}.$$

This completes the proof. □

Theorem 4.12. *Let $\{\mathbf{E}_i\}_{i\in\Lambda}$ be a family of neutrosophic sets on X. Define*

$$\mathbf{T}(\mathbf{x}) = \inf_{i\in\Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}),\ \mathbf{I}(\mathbf{x}) = \inf_{i\in\Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}),\ \textit{and}\ \mathbf{F}(\mathbf{x}) = \sup_{i\in\Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}).$$

Then, $\left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)} = \bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)}$.

Proof. Let $\mathbf{x} \in \left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)}$. Then,

$$\inf_{i\in\Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha,\ \inf_{i\in\Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta,\ \text{and}\ \sup_{i\in\Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Consequently,

$$\mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha,\ \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta,\ \text{and}\ \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma$$

for every $i \in \Lambda$. Thus,

$$\mathbf{x} \in (\mathbf{E}_i)_{(\alpha,\beta,\gamma)}$$

for every $i \in \Lambda$, and therefore,

$$\mathbf{x} \in \bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)}.$$

Hence,

$$\left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)} \subseteq \bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)}.$$

Conversely, if

$$\mathbf{x} \in \bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)},$$

then for every $i \in \Lambda$,

$$\mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha,\quad \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta,\quad \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Therefore,

$$\inf_{i\in\Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha,\ \inf_{i\in\Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta,\ \text{and}\ \sup_{i\in\Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Hence,

$$\mathbf{x} \in \left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)}.$$

Thus,

$$\bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)} \subseteq \left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)}.$$

Therefore,

$$\left(\bigcap_{i\in\Lambda} \mathbf{E}_i\right)_{(\alpha,\beta,\gamma)} = \bigcap_{i\in\Lambda} (\mathbf{E}_i)_{(\alpha,\beta,\gamma)}.$$

□

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